4.6 Related Rates

I. Related Rates

A. Procedure

1) Sketch a picture
2) List known rates and values. Identify the rate you want to find.
3) Write an equation.
4) Implicitly differentiate with respect to time.
5) Substitute in known values.
6) Solve and answer the question

B. Cautions

1) Always differentiate with respect to time
2) Only quantities that don't change can be substituted in at the beginning. Quantities that change over time must be substituted in after differentiating.
3) Increasing rates are positive. Decreasing rates are negative. However, if it asks "at what rate is x decreasing?", the negative is already implied.

C. Examples

1) A spherical balloon is inflated with helium at the rate of 100π ft³/min.

a) How fast is the balloon's radius increasing at the instant the radius is 5 ft?

$\frac{dV}{dt} = 100\pi \quad r = 5 \quad$ Find $\frac{dr}{dt}$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt}$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$100\pi = 4\pi(5)^2 \frac{dr}{dt}$

$\boxed{\frac{dr}{dt} = 1 \text{ ft/min}}$

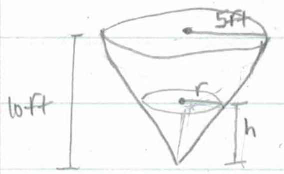
b) How fast is the surface area increasing at this moment? (Find $\frac{ds}{dt}$)

$$S = 4\pi r^2$$
$$\frac{ds}{dt} = 4\pi \left(2r \frac{dr}{dt}\right)$$
$$= 8\pi r \frac{dr}{dt}$$

(From a) $r = 5$ $\frac{dr}{dt} = 1$)

$$= 8\pi(5)(1)$$
$$\boxed{\frac{ds}{dt} = 40\pi \text{ ft}^2/\text{min}}$$

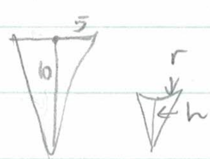
2) Water runs into a conical tank at the rate of 9 ft³/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6ft deep?



$\frac{dV}{dt} = 9$ $h = 6$ Find $\frac{dh}{dt}$

Method 1: Eliminate r and then differentiate

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$$
$$= \frac{1}{3}\pi \left(\frac{h^2}{4}\right) h$$
$$V = \frac{1}{12}\pi h^3$$
$$\frac{dV}{dt} = \frac{1}{12}\pi \left(3h^2 \frac{dh}{dt}\right)$$
$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$
$$9 = \frac{1}{4}\pi (6)^2 \frac{dh}{dt}$$
$$9 = \frac{36}{4}\pi \frac{dh}{dt}$$
$$\boxed{\frac{dh}{dt} = \frac{1}{\pi} = 0.3183 \text{ ft/min}}$$



To get r in terms of h:
Use proportions:

$$\frac{h}{r} = \frac{10}{5}$$
$$10r = 5h$$
$$r = \frac{h}{2}$$
$$\frac{dr}{dt} = \frac{1}{2}\frac{dh}{dt}$$

Method 2: Differentiate w/ 3 variables

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt} + h\left(\frac{1}{3}\pi\, 2r \frac{dr}{dt}\right)$$

$$\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt} + \frac{2}{3}\pi r h \frac{dr}{dt}$$

$$9 = \frac{1}{3}\pi\left(\frac{6}{2}\right)^2 \frac{dh}{dt} + \frac{2}{3}\pi\left(\frac{6}{2}\right)(6)\left(\frac{1}{2}\frac{dh}{dt}\right)$$

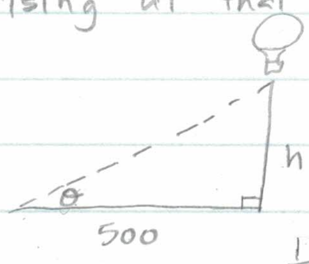
$$9 = \frac{9}{3}\pi \frac{dh}{dt} + \frac{18}{3}\pi \frac{dh}{dt}$$

$$9 = 3\pi \frac{dh}{dt} + 6\pi \frac{dh}{dt}$$

$$9 = 9\pi \frac{dh}{dt}$$

$$\boxed{\frac{dh}{dt} = \frac{1}{\pi} = 0.3183 \text{ ft/min}}$$

3) A hot-air balloon rising straight up from a level field is tracked by a range finder 500 ft from the lift off point. At the moment the range finder's elevation angle is $\frac{\pi}{4}$, the angle increasing at a rate of 0.14 radians per minute. How fast is the balloon rising at that moment?



$\theta = \frac{\pi}{4}$

$\frac{d\theta}{dt} = 0.14$

Find $\frac{dh}{dt}$

$\frac{1}{500}h$

$\tan\theta = \frac{h}{500}$

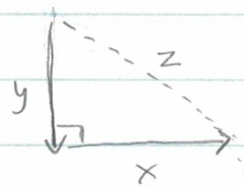
$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{500}\frac{dh}{dt}$

$[\sec(\frac{\pi}{4})]^2 \rightarrow \sec^2(\frac{\pi}{4})(0.14) = \frac{1}{500}\frac{dh}{dt}$

$500\left(\frac{2}{\sqrt{2}}\right)^2(0.14) = \frac{dh}{dt}$

$\boxed{\frac{dh}{dt} = 140 \text{ ft/min}}$

4) A police cruiser approaching a right angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determined with radar that the distance between them and the car is increasing 20 mph.
If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?



$y = 0.6 \quad x = 0.8 \quad z = 1 \quad 0.6^2 + 0.8^2 = z^2$

$\frac{dy}{dt} = -60 \quad \frac{dx}{dt} = ? \quad \frac{dz}{dt} = 20 \quad .36 + .64 = z^2$

$$x^2 + y^2 = z^2$$

derivative

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$2(0.8)\frac{dx}{dt} + 2(0.6)(-60) = 2(1)(20)$$

$$1.6\frac{dx}{dt} = 112$$

$$\boxed{\frac{dx}{dt} = 70\text{mph}}$$